VEKTORANALYS

Kursvecka 5

övningar

PROBLEM 1

Calculate in spherical coordinates: $\nabla\left(\nabla\cdot\hat{e}_r\right)-\nabla\times\left(\nabla\times\hat{e}_r\right)$

NOTE: as we will see next week $\nabla\left(\nabla\cdot\hat{e}_r\right)-\nabla\times\left(\nabla\times\hat{e}_r\right)=\nabla^2\hat{e}_r$

SOLUTION

$$\nabla\left(\nabla\cdot\hat{e}_r\right)-\nabla\times\left(\nabla\times\hat{e}_r\right)$$

We need gradient, divergence and curl in spherical coordinates

SOLUTION

Spherical coord: (r,θ,φ)

$$\bar{r} = \left(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta\right)$$

$$\begin{cases} h_r = 1 \\ h_\theta = r \\ h_\varphi = r\sin\theta \end{cases}$$

$$grad\phi = \left(\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\right) = \left(\frac{1}{h_r}\frac{\partial\phi}{\partial r}, \frac{1}{h_\theta}\frac{\partial\phi}{\partial\theta}, \frac{1}{h_\varphi}\frac{\partial\phi}{\partial\varphi}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\boxed{grad\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\right)}$$

$$div\bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(A_1h_2h_3\right) + \frac{\partial}{\partial u_2}\left(A_2h_3h_1\right) + \frac{\partial}{\partial u_3}\left(A_3h_1h_2\right)\right] =$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(A_r r^2\sin\theta\right) + \frac{\partial}{\partial\theta}\left(A_\theta r\sin\theta\right) + \frac{\partial}{\partial\varphi}\left(A_\varphi r\right)\right]$$

$$\boxed{div\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)}$$

SOLUTION

$$rot\bar{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

$$rot\bar{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right), \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \right)$$

Let’s go back to our initial problem:

$$\nabla^2 \hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right)$$

$$\hat{e}_r = (1,0,0)$$ *(in the spherical coordinate system!)*

So we have to calculate:

$$div\bar{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)$$

$$with\ \bar{A} = \hat{e}_r = (1,0,0)$$

SOLUTION

Therefore: $div\,\hat{e}_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right) = \frac{2}{r}$

Then we have to calculate:

$$rot\bar{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right), \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \right)$$

$with\ \bar{A} = \hat{e}_r = (1, 0, 0)$

Therefore: $rot\,\hat{e}_r = \left(0, 0, 0\right)$

$$grad\,\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi} \right)$$

Finally:

$$\nabla^2\hat{e}_r = grad\left(div\,\hat{e}_r\right) - rot\left(rot\,\hat{e}_r\right) = grad\left(\frac{2}{r}\right) - \nabla\times(0,0,0) = \left(-\frac{2}{r^2}, 0, 0\right) = -\frac{2}{r^2}\hat{e}_r$$

PROBLEM 2

Consider the following fields: $\psi = -\frac{\cos\theta}{r^2}, \qquad \bar{A} = \frac{\sin\theta}{r^2}\hat{e}_\varphi$

where (r,θ,φ) are spherical coordinates.
Calculate:

$a) \quad \nabla\psi$

$b) \quad \nabla\times\bar{A}$

$c) \quad \nabla^2\psi \quad and \quad \nabla\times\left(\nabla\times\bar{A}\right)$

SOLUTION

(a) $$grad\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right) = \left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}, 0\right) = \frac{2\cos\theta}{r^3}\hat{e}_r + \frac{\sin\theta}{r^3}\hat{e}_\theta$$

(b) $$rot\bar{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\varphi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right), \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

$$with\ \bar{A} = \left(0, 0, \frac{\sin\theta}{r^2}\right)$$

SOLUTION

$$rot\bar{A} = \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right), -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right), 0 \right) = \left(\frac{1}{r^3\sin\theta} 2\sin\theta\cos\theta, \frac{\sin\theta}{r}\frac{1}{r^2} \right) =$$

$$\left(\frac{2\cos\theta}{r^3}, \frac{\sin\theta}{r^3}, 0 \right) = grad\psi$$

(c)

$$\nabla^2\psi = \nabla\cdot(\nabla\psi) = \nabla\cdot(\nabla\times\bar{A}) = 0$$

$$\nabla\times(\nabla\times\bar{A}) = \nabla\times(\nabla\psi) = 0$$

see ID7 and ID8 from week 5

PROBLEM 3

The Orion nebula can be considered as a sphere with constant radius R_0.
Assume that the velocity of the Helium atoms in the nebula is described by the following field:

$$\bar{v} = k\left(r - \frac{r^2}{2}\cos\varphi\right)\hat{e}_r + k\left(2r^2\sin\theta\sin\varphi\right)\hat{e}_\varphi$$

Calculate the flux of Helium atoms that flows outside the nebula

SOLUTION

Guass theorem

$$Flux = \iint_S \bar{v}\cdot d\bar{S} = \iiint_V div(\bar{v})dV$$

$$\nabla\cdot\bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(v_\varphi\right)$$

(in a spherical coord. system)

$$div\left(k\left(r - \frac{r^2}{2}\cos\varphi\right), 0, 2r^2k\sin\theta\sin\varphi\right) = k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 - \frac{r^4}{2}\cos\varphi\right) + k\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(2r^2\sin\theta\sin\varphi\right) =$$

$$= \frac{k}{r^2}\left(3r^2 - 4\frac{r^3}{2}\cos\varphi\right) + k_2\frac{2r^2\sin\theta}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(\sin\varphi\right) = 3k - 2kr\cos\varphi + 2kr\cos\varphi = 3k$$

$$\iint_S \bar{v}\cdot d\bar{S} = \iiint_V 3k dV = 3\frac{4\pi}{3}R_0^3 = 4\pi k R_0^3$$